

X-550-70-85

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# FIRST ORDER PERTURBATIONS OF AN ORBIT BY A MASS ANOMALY

DAVID E. SMITH

MARCH 1970



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**GODDARD SPACE FLIGHT CENTER**

**GREENBELT, MARYLAND**

FACILITY FORM 602

N70-28589  
(ACCESSION NUMBER)

21  
(PAGES)

TMX-63921  
(NASA CR OR TMX OR AD NUMBER)

(THRU)

(CODE)

30  
(CATEGORY)

X-550-70-85

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A MASS ANOMALY

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ABSTRACT

The first order short period perturbations of a satellites position and the first order long period perturbations of the orbital elements by a mass anomaly (mascon) are developed.

# FIRST ORDER PERTURBATIONS OF AN ORBIT BY A MASS ANOMALY

## INTRODUCTION

Mass concentrations on the moon have been discovered from the analysis of the orbits of lunar satellites (see, for example, references 1, 2, 3) and these mascons, as they are usually called, have made the determination of orbits of lunar satellites difficult and their prediction rather unreliable.

The effect of a mass concentration is, in certain respects, similar to that of a third body except that the mass concentration is entirely inside the orbit of the spacecraft and is in a fixed relationship with respect to the primary body. This similarity, however, enables the perturbing effects of a mascon to be calculated from a very restricted three body approach as might be employed, for example, in computing the first order effect of the moon on a near Earth satellite.

This is the approach that has been followed here. The disturbing function of a mascon has been obtained and used to develop the first order short and long period perturbations of the orbit of a spacecraft.

## DISTURBING POTENTIAL OF A MASCON

Let us consider the potential at the point  $P(x, y, z)$  due to a mascon,  $m$ , at  $(X, Y, Z)$  and another mass  $(M - m)$  at  $(x_1, y_1, z_1)$ . Let the origin of the coordinate system be at the center of gravity of the two masses (see Figure 1) and let the distances of  $P$  from  $m$ , the origin and  $(M - m)$  be  $\rho, r, \rho_1$ , respectively. The potential ( $V$ ) at  $P$  can therefore be written.

$$V = \frac{G (M - m)}{r_1} + \frac{G m}{\rho} \quad (1)$$

Further, because the two masses  $m$  and  $(M - m)$  are in a fixed relationship, the disturbing potential is the difference between the potential at  $P$  and the potential of the central force term alone at  $P$ . Now the potential of the central force term is

$$\frac{GM}{r}$$

and hence the disturbing potential ( $\Delta V$ ) can be written as

$$\begin{aligned} \Delta V &= V - \frac{GM}{r} \\ &= G \left( \frac{M - m}{r_1} + \frac{Gm}{\rho} - \frac{GM}{r} \right) \end{aligned} \quad (2)$$

From the evaluation so far undertaken the size of a mascon is of the order of  $10^{-5}$  of the moon's mass so we can justifiably neglect terms of order  $(m/M)^2$ . In addition, because the origin of the coordinate system is at the center of gravity, we have

$$-\frac{r_1}{R} = \frac{x_1}{X} = \frac{y_1}{Y} \sim -\frac{m}{M} + \text{terms of order} \left( \frac{m}{M} \right)^2 \quad (3)$$



and

$$\frac{r_1}{r} < \frac{m}{M} \quad (4)$$

where  $r_1$  is the distance of  $(M - m)$  from the origin. Therefore, neglecting terms of order  $(m/M)^2$  we have from Figure 1

$$\frac{1}{r_1} \approx \frac{1}{r} \left(1 - \frac{r_1}{r} \cos \phi\right) \quad (5)$$

and equation 2 can be written

$$\Delta V = Gm \left[ \frac{1}{r} - \frac{1}{r} - \frac{R \cos \phi}{r^2} \right] \quad (6)$$

In addition, from Figure 1, we have

$$\cos \phi = \frac{xX + yY + zZ}{rR} \quad (7)$$

and therefore the disturbing potential of a mascon can be written

$$\begin{aligned} \Delta V = Gm \left[ \frac{1}{r} - \frac{1}{r} - \frac{(xX + yY + zZ)}{r^3} \right] \\ + \text{terms of order} \left( \frac{m}{M} \right)^2 \end{aligned} \quad (8)$$

From the disturbing potential we can obtain the disturbing acceleration, of which the x-component is

$$\frac{\partial}{\partial x} (\Delta V) = -Gm \left[ \left( \frac{1}{\rho^3} - \frac{1}{r^3} \right) (x - X) - \frac{3x}{r^5} (xX + yY + zZ) \right] \quad (9)$$

with similar expressions for the y and z components. In order to develop the perturbations to the elements we require the components of the disturbing acceleration with respect to the orbit. Let the component acting along the radius vector be  $\bar{R}$ , the component acting perpendicularly to the orbit plane  $\bar{S}$  and the component in the orbit plane acting perpendicularly to  $\bar{R}$  be  $\bar{Q}$ , then

$$\bar{R} = \ell_1 \frac{\partial}{\partial x} (\Delta V) + m_1 \frac{\partial}{\partial y} (\Delta V) + n_1 \frac{\partial}{\partial z} (\Delta V) \quad (10)$$

$$\bar{S} = \ell_2 \frac{\partial}{\partial x} (\Delta V) + m_2 \frac{\partial}{\partial y} (\Delta V) + n_2 \frac{\partial}{\partial z} (\Delta V) \quad (11)$$

$$\bar{Q} = \ell_3 \frac{\partial}{\partial x} (\Delta V) + m_3 \frac{\partial}{\partial y} (\Delta V) + n_3 \frac{\partial}{\partial z} (\Delta V) \quad (12)$$

where

$$(\ell_1, m_1, n_1), (\ell_2, m_2, n_2) \text{ and } (\ell_3, m_3, n_3)$$

are the direction cosines of  $\bar{R}$ ,  $\bar{Q}$  and  $\bar{S}$ .

From Figure 2 we have, following the method of Cook (reference 4),

$$\ell_1 = \frac{x}{r}, m_1 = \frac{y}{r}, n_1 = \frac{z}{r} \quad (13)$$

$$\ell_2 = \cos \Omega \sin u - \sin \Omega \cos u \cos i$$

$$\begin{aligned}
m_2 &= -\sin \Omega \sin u + \cos \Omega \cos u \cos i \\
n_2 &= \cos u \sin i
\end{aligned} \tag{14}$$

$$\begin{aligned}
l_3 &= \sin \Omega \sin i \\
m_3 &= -\cos \Omega \sin i
\end{aligned} \tag{15}$$

$$n_3 = \cos i$$

where

$\Omega$  is the right ascension of the node

$u$  is the argument of latitude

$i$  is the orbital inclination

If the position of the mascon is right ascension  $\alpha$  and declination  $\delta$ , then we also have

$$X = R \cos \delta \cos \alpha$$

$$Y = R \cos \delta \sin \alpha$$

$$Z = R \sin \delta$$

The distance  $\rho$  of the satellite from the mascon can be obtained in terms of  $r$ ,  $R$  and  $\phi$  using the cosine rule which, upon expansion, can be written as

$$\frac{1}{p^3} = \frac{1}{r^3} \left[ 1 + 3 \frac{R}{r} \cos \phi - \frac{3}{2} \left( \frac{R}{r} \right)^2 (1 - 5 \cos^2 \phi) - - - \right] \quad (17)$$

With the aid of equations 9 through 17 we can write the major terms in the components of the disturbing acceleration as follows:

$$\bar{R} = \frac{Gm}{r^3} \left( \frac{3R^2}{2r} \right) \left[ 1 - \frac{3}{2} (A^2 + B^2) - 3 A B \sin 2 u - \frac{3}{2} (A^2 - B^2) \cos 2 u \right] \quad (18)$$

$$\bar{Q} = \frac{Gm}{r^3} \left( \frac{3R^2}{r} \right) \left[ A B \cos 2u - \frac{1}{2} (A^2 - B^2) \sin 2 u \right] \quad (19)$$

$$\bar{S} = \frac{Gm}{r^3} \left( \frac{3R^2}{r} \right) C [A \cos u + B \sin u] \quad (20)$$

where

$$A = \cos \delta \cos (\alpha - \Omega) \quad (21)$$

$$B = \sin i \sin \delta + \cos i \cos \delta \sin (\alpha - \Omega) \quad (22)$$

$$C = \cos i \sin \delta - \sin i \cos \delta \sin (\alpha - \Omega) \quad (23)$$

The notation used above is the same as that used by Cook in reference 4 where details of the method employed here may be found. The terms neglected in equations 18, 19 and 20 correspond to terms of order  $(R/r)^3$  and above in equation 17.

## PERTURBATION EQUATIONS

The perturbations to the elliptic elements as a result of the disturbing acceleration are given by Lagrange's Planetary Equations. One form of these equations (references 4 and 5) accurate to order  $(m/M)$ , is

$$\frac{da}{d\theta} = \frac{2r^2}{n^2 a p} \left[ \bar{R} e \sin \theta + \frac{p}{r} \bar{Q} \right] \quad (24)$$

$$\frac{de}{d\theta} = \frac{r^2}{n^2 a^3} \left[ \bar{R} \sin \theta + \bar{Q} (\cos \theta + \cos E) \right] \quad (25)$$

$$\frac{d\Omega}{d\theta} = \frac{\bar{S} r^3 \sin u}{n^2 a^3 p \sin i} \quad (26)$$

$$\frac{di}{d\theta} = \frac{\bar{S} r^3 \cos u}{n^2 a^3 p} \quad (27)$$

$$\frac{d\omega}{d\theta} = \frac{r^2}{n^2 a^3 e} \left[ -\bar{R} \cos \theta + \left(1 + \frac{r}{p}\right) \bar{Q} \sin \theta \right] - \frac{d\Omega}{d\theta} \cos i \quad (28)$$

$$\frac{d\sigma}{d\theta} = -\frac{2 r^3 \bar{R}}{n^2 a^4 (1 - e^2)^{1/2}} \quad (29)$$

$$- (1 - e^2)^{1/2} \left[ \frac{d\omega}{d\theta} + \cos i \frac{d\Omega}{d\theta} \right]$$

where

$$\sigma = M - \int n dt$$

$a$  is the semi-major axis

$e$  is the eccentricity

$\omega$  is the argument of perigee

$M$  is the mean anomaly

$E$  is the eccentric anomaly

$\bar{e}$  is the true anomaly

$n$  is the mean motion

and

$$p = a (1 - e^2)$$

In addition to the Keplerian elements it is of interest to know the perturbations to radial distance ( $r$ ), position in the orbit and direction of travel. The latter is effectively given by the short-period perturbations of  $i$  for mascons near the equator but the perturbation equations for  $r$  and position in the orbit need to be derived.

The radial distance is given by

$$r = \frac{p}{1 + e \cos \theta} \quad (30)$$

and hence

$$\frac{dr}{d\vartheta} - \frac{er^2}{p} \sin \vartheta = \frac{r}{a} \frac{da}{d\vartheta} - \frac{r}{p} (2ae + r \cos \vartheta) \frac{de}{d\vartheta} \quad (31)$$

Substituting for  $da/d\vartheta$  and  $de/d\vartheta$  from equations 24 and 25 and simplifying leads to

$$\frac{dr}{d\vartheta} - \frac{er^2 \sin \vartheta}{p} = \frac{r^4}{n^2 a^3 p} \left[ \sin \theta \cos \theta \bar{R} + \left( 2 + \frac{r \cos \theta}{ae} - \frac{p \cos \vartheta}{re} \right) Q \right] \quad (32)$$

for the instantaneous perturbations of radial distance.

The mean anomaly  $M$  has been defined as

$$M = \sigma + \int n dt \quad (33)$$

and hence

$$\begin{aligned} \frac{dM}{d\vartheta} &= \frac{d\sigma}{d\vartheta} + \int \frac{dn}{d\vartheta} d\vartheta \\ &= \frac{d\sigma}{d\vartheta} + \frac{3}{2a^3 (1-e^2)^{1/2}} \int r^2 \frac{da}{d\vartheta} d\vartheta \\ &= \frac{d\sigma}{d\vartheta} - \frac{3(1-e^2)^{1/2}}{n^2 a^3 p^2} \int r^4 \left( \bar{R} e \sin \theta + \frac{p}{r} \bar{Q} \right) d\vartheta \end{aligned} \quad (34)$$

Substituting for  $d\sigma/d\vartheta$  from equation 29 leads to

$$\begin{aligned} & \frac{dM}{d\vartheta} + (1 - e^2)^{1/2} \left[ \frac{d\omega}{d\vartheta} + \cos i \frac{d\Omega}{d\vartheta} \right] \\ &= -\frac{2r^3}{n^2 a^3 p} (1 - e^2)^{1/2} \cdot \bar{R} - \frac{3(1 - e^2)^{1/2}}{n^2 a^3 p^2} \int r^4 \left( \bar{R} e \sin \theta + \frac{p}{r} \bar{Q} \right) d\vartheta \end{aligned} \quad (35)$$

where the left hand side is effectively the perturbation to the along track position.

#### SHORT PERIOD PERTURBATIONS

Substituting for  $\bar{R}$  and  $\bar{Q}$  in equation (32) for the perturbations to the radial distance and integrating leads to

$$\begin{aligned} \delta r = & -\frac{3}{2} \left( \frac{m}{M} \right) \left( \frac{R^2}{p} \right) \left[ \frac{1}{4} \left\{ 1 - \frac{3}{2} (A^2 + B^2) \right\} \cos 2\theta \right. \\ & - \left\{ A B \sin 2u + \frac{1}{2} (A^2 - B^2) \cos 2u \right\} \\ & + \left\{ A B \sin 2\omega + \frac{1}{2} (A^2 - B^2) \cos 2\omega \right\} \left( \frac{3}{4} + \frac{1}{16} \cos 4\theta + \frac{4e}{3} \cos^3 \theta - \frac{4e}{5} \cos^5 \theta \right) \\ & + \left\{ A B \cos 2\omega - \frac{1}{2} (A^2 - B^2) \sin 2\omega \right\} \left( \frac{7}{4} \theta + \frac{1}{16} \sin 4\theta + \frac{2e}{3} \sin^3 \theta - \frac{4e}{5} \sin^5 \theta \right) \\ & \left. + \text{terms of order } e^2 \right] \end{aligned} \quad (36)$$



where  $\delta r$  is the radial perturbation of  $r$ .

Similarly, substituting for  $\bar{R}$  and  $\bar{Q}$  in equation 35 and integrating twice, where necessary, leads to

$$\begin{aligned}
& \delta M + (1 - e^2)^{1/2} \left[ \delta \omega + \cos i \delta \Omega \right] \\
&= -3 \left( \frac{m}{M} \right) \left( \frac{R}{P} \right)^2 (1 - e^2)^{1/2} \left[ \left\{ 1 - \frac{3}{2} (A^2 + B^2) \right\} \left( \theta - \frac{1}{2} e \sin \theta \right) \right. \\
&\quad + \frac{3}{2} \left\{ A B \cos 2u - \frac{1}{2} (A^2 - B^2) \sin 2u \right\} \\
&\quad - e \left\{ A B \sin 2\omega + \frac{1}{2} (A^2 - B^2) \cos 2\omega \right\} \left( \frac{5}{2} \sin \theta - \frac{1}{3} \sin^3 \theta \right) \\
&\quad \left. + e \left\{ A B \cos 2\omega - \frac{1}{2} (A^2 - B^2) \sin 2\omega \right\} \left( 2 \cos \theta + \frac{1}{3} \cos^3 \theta \right) \right] \quad (37)
\end{aligned}$$

The change in direction of travel of a spacecraft can be represented by the change in orbital inclination. Substituting for  $\bar{S}$  in equation 27 and integrating gives

$$\begin{aligned}
\delta i &= \frac{3}{2} \left( \frac{m}{M} \right) \left( \frac{R}{P} \right)^2 C \left[ A \theta + \frac{1}{2} (A \sin 2u - B \cos 2u) \right. \\
&\quad + e \sin \theta \left( 1 - \frac{2}{3} \sin^2 \theta \right) (A \cos 2\omega + B \sin 2\omega) \\
&\quad \left. + \frac{3}{2} e \cos^3 \theta (A \sin 2\omega - B \cos 2\omega) \right] \quad (38)
\end{aligned}$$

Equations 36, 37 and 38 give the first order perturbations of the radial and along track positions and the direction of travel of a satellite. The maximum numerical size of the short periodic terms (those containing  $u$  or  $v$ ) for a near lunar satellite can be seen to be of the following order for a mascon of  $10^{-5}$  of the moon's mass:

$$\delta r \sim -\frac{3}{4} \left( \frac{m}{M} \right) \left( \frac{R^2}{p} \right) \sim 15 \text{ meters}$$

$$\delta M + \delta \omega + \cos i \delta \Omega \sim \frac{9}{4} \left( \frac{m}{M} \right) \left( \frac{R}{p} \right)^2 \sim 4 \text{ arc seconds}$$

$$\delta i \sim \frac{3}{8} \left( \frac{m}{M} \right) \left( \frac{R}{p} \right)^2 \sim 0.7 \text{ arc seconds}$$

#### LONG PERIOD AND SECULAR PERTURBATIONS

Substituting for  $R$ ,  $S$  and  $Q$  in equations 24, 25, 26 and 28 and integrating from 0 to  $2\pi$  with respect to  $\theta$ , leads to

$$\Delta a = 0 \tag{39}$$

$$\Delta e = 0 \tag{40}$$

$$\Delta \Omega = 3\pi \left( \frac{m}{M} \right) \left( \frac{R}{p} \right)^2 B C / \sin i \tag{41}$$

$$\Delta \omega + \cos i \Delta \Omega = -3\pi \left( \frac{m}{M} \right) \left( \frac{R}{p} \right)^2 \left[ 1 - \frac{3}{2} (A^2 + B^2) \right] \tag{42}$$

where the  $\Delta$  quantities represent the change in the parameters per revolution of the satellite.

The long period perturbation of the inclination can be obtained from equation 38 by neglecting terms in sine and cosine  $\hat{\theta}$  and replacing  $\hat{\theta}$  by  $2\pi$ . We therefore have

$$\Delta i = 3\pi \left(\frac{m}{M}\right) \left(\frac{R}{P}\right)^2 AC \quad (43)$$

Similarly, the long period and secular perturbations of the along track position can be obtained from equation 37. Hence

$$\begin{aligned} & \Delta M + (1 - e^2)^{1/2} \left[ \Delta \omega + \cos i \Delta \Omega \right] \\ &= -6\pi \left(\frac{m}{M}\right) \left(\frac{R}{P}\right)^2 (1 - e^2)^{1/2} \left[ 1 - \frac{3}{2} (A^2 + B^2) \right] \end{aligned} \quad (44)$$

Because there are no long period or secular perturbations of the semi-major axis (equation 39) we have

$$\Delta \sigma = \Delta M$$

and hence equation 44 could have been obtained from equation 29 directly.

For a close lunar satellite and  $(m/M) = 10^{-5}$  the maximum magnitude of the first order long period and secular perturbations are

$$\Delta\dot{\Omega} \sim 8' \sin i \text{ arc seconds/revolution}$$

$$\Delta\alpha + \cos i \Delta\Omega \sim 16 \text{ arc seconds/revolution}$$

$$\Delta i \sim 8 \text{ arc seconds/revolution}$$

$$\Delta M + \Delta\alpha + \cos i \Delta\Omega \sim 32 \text{ arc seconds/revolution}$$

## CONCLUSIONS

The first order perturbations of an orbit by a mass concentration have been developed and it has been shown that the maximum values of the secular and long period perturbations are about 300 and 80 meters in the along track and across track directions (1 arc second on the moon's surface is about 10 meters) per revolution for a mascon of  $10^{-5}$  of the moon's mass. In addition, short period perturbations have been found with maximum amplitudes of about 15 meters in height, 40 meters in along track position and 7 meters in the across track direction.

The major assumptions that were made in the analysis were that the position of the mascon and the orbit of the spacecraft remained fixed in space over the integration period (one revolution). The former approximation can be justified because the moon rotates very slowly (about 0.5 degrees) per hour) and the second approximation is valid if the perturbations by all other forces are small, which is usually the case.

The main restriction on the whole theory is that it is only of first order. Neglecting higher powers of  $(R/r)$  in the expansion of the disturbing acceleration limits the region of applicability of the theory. For example, if the mascon is near the surface of the body ( $R$  comparable to the radius of the body) then the errors in the acceleration are of order 10% for  $r \sim 10R$ . However, if the mascon is near the center of the main body, the theory is valid for even very close orbits and should indicate the true changes in the orbit. One further restriction is that the theory cannot be applied to circular orbits. The integrations performed were with respect to the true anomaly thus implying that a perigee could be defined and therefore the orbit was non-circular.

In addition to the application of the equations that have been derived to certain orbital computation problems, the analysis enables the approximate magnitude and character of mascon perturbations to be easily assessed, which could be useful in preliminary studies of orbit problems.

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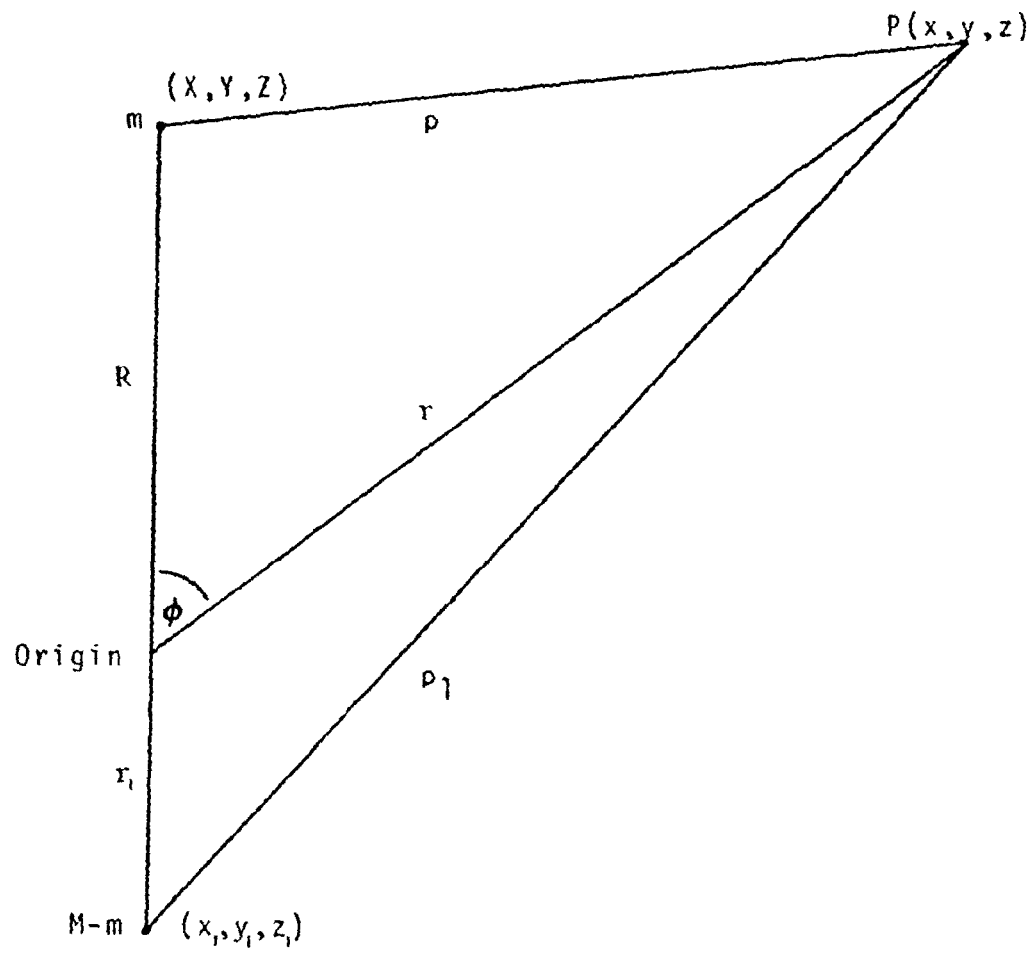


Figure 1. Coordinate System for Point Mass

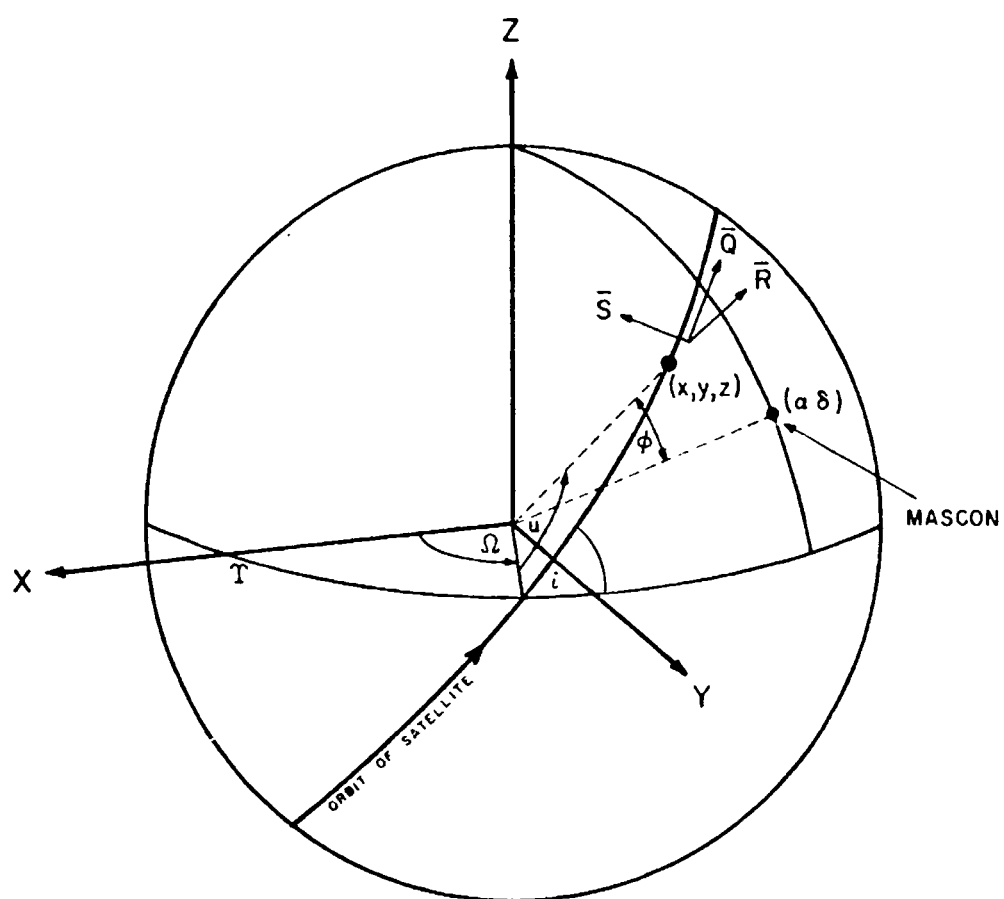


Figure 2. Satellite Orbit, Mascon and Components of Disturbing Acceleration